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STOCHASTIC SENSITIVITY ANALYSIS OF CYLINDRICAL SHELL

Abstract

The paper deals with some chosen aspects of stochastic sensitivity structural analysis and its application in the engineering practice. The main aim of the study is to provide the generalized stochastic perturbation technique based on classical Taylor expansion with a single random variable. The study is illustrated by numerical results concerning an industrial thin shell structure modeled as a 3-D structure.

Keywords

Sensitivity, stochastic perturbation technique, finite element method, shell structure.

1 INTRODUCTION

It is known that the most effective computational tool in civil engineering nowadays is the finite element method [1, 2], as the basis of almost all structural analysis computer codes. On the other hand, in modern design, sensitivity analysis cannot be avoided, since it makes it possible to determine the so-called starting point (or design point), leading to the optimal solution. Recently, the sensitivity issues are discussed extensively in the literature. Background of the design sensitivity analysis is presented in [3], for instance. The sensitivity analysis can be carried out with respect to local design variables, such as cross-sectional area, element thickness, Young's modulus, Poisson's ratio, loading [4].

In accordance with developments of the computational technique, uncertainties of the design variables appear to be necessarily needed in the state-of-the-art methodologies in computer terms. Besides the traditional Monte Carlo simulation, we may mention the spectral approach [5] and, seeming more effective, perturbation approach [6, 7]. In the latter, all the functions of random variables are expanded exponentially. By using the first two probabilistic moments for random variables on input, the first two probabilistic moments of the structural response are obtained on output; the expectations are second-order accurate, while the cross-covariances are first-order accurate [8-10, 14].

In paper a new problem of computational mechanics is formulated for static stochastic sensitivity (before problem was solve separately [11] and [12]). Since both the random and design variables are expressed in a dicretized-parameter space, the stochastic sensitivity function can be modelled in a parallel way and evaluated by using a conventional deterministic finite element technique; and the computer procedures can be carried out in parallel for dual systems and sequentially for their 0th-, 1th- and 2nd-order equations.

2 FINITE ELEMENT FORMULATION

Consider structural response of the linear-elastic systems with N degrees of freedom described by the functional

$$\Phi = G[q(\boldsymbol{h},\boldsymbol{b}),\boldsymbol{b}] \tag{1}$$

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and the system equilibrium equations

$$K_{\alpha\beta}(\boldsymbol{h},\boldsymbol{b})\ q_{\beta}(\boldsymbol{h},\boldsymbol{b}) = f_{\alpha}(\boldsymbol{h},\boldsymbol{b}) \qquad \alpha,\beta = 1,\dots,N \qquad (2)$$

where $\boldsymbol{h} = \{h^d\},\ d = 1,\dots,D$ and $\boldsymbol{b} = \{b^r\},\ r = 1,\dots,R$, are vector of design variables and random variables. Some of all components in the vector $\boldsymbol{h}$ and $\boldsymbol{b}$ can coincide. Symbols $K_{\alpha\beta}(\boldsymbol{h},\boldsymbol{b}), f_{\alpha}(\boldsymbol{h},\boldsymbol{b})$ and $q_{\beta}(\boldsymbol{h},\boldsymbol{b})$ represent the stiffness matrix, the vectors of external loads and nodal displacements, respectively.

Since uncertainties in geometry and material properties of the system member are taken on account, the stiffness, load and displacement are random functions; $K_{\alpha\beta}(\boldsymbol{h},\boldsymbol{b})$ and $f_{\alpha}(\boldsymbol{h},\boldsymbol{b})$ are generally explicit functions of random and design variables, whereas $q_{\beta}(\boldsymbol{h},\boldsymbol{b})$ is implicit function of these variables.

The random variables b^r can be defined via the first two probabilistic characteristics: expectations $\overline{b^r}$ and cross-covariances $Cov(b^r,b^s)$ as

$$\overline{b^r} = E[\,b^r] = \int_{-\infty}^{+\infty} b^r\,p(b^r)\,db^r \qquad (3)$$

$$Cov(b^r,b^s) = E[\,(b^r - b_0^r)(b^s - b_0^s)] = \\ = R(b^r,b^s)\sqrt{Var(b^r)Var(b^s)} \qquad (4)$$

with

$$R(b^r,b^s) = \iint_{-\infty}^{+\infty} b^r\,b^s\ p(b^r,b^s)db^r db^s \qquad r,s = 1,\dots,R \qquad (5)$$

$$Var(b^r) = \alpha^2 E^2\,[\,b^r] \qquad (6)$$

where $R(b^r,b^s)$, $Var(b^r,b^s)$, $p(b^r,b^s)$ and α denote: functions of correlation, variance, joint probability density and the coefficient of variation, respectively.

The functions of random variables $K_{\alpha\beta}$, f_{α} and q_{β} can be handled with the finite difference technique or by the leas square fit method, cf. [7], for instance. In this paper, a perturbation scheme will be employed.

Suppose that , f_{α} and q_{β} are twice differentiable with respect to h^d. Using the chain rule of differentiation leads to

$$\Phi^{;d} = \Phi^{,d} + \Phi_{,\alpha} q_{\alpha}^{;d} \qquad (7)$$

where $(\cdot)^{;d}$ is the first ordinary derivative with respect to the dth design variable, while $(\cdot)^{,d}$ and $(\cdot)_{,\alpha}$ are the first partial derivatives with respect to the d-th design variable and α-th nodal displacement, respectively.

Because Φ is an explicit function of its arguments, the components $\Phi^{,d}$ and $\Phi_{,\alpha}$ are known. The derivatives $q_{\alpha}^{;d}$ must be determined, since q_{α} are implicit with respect to h^d. Differentiating the equilibrium Eq. (2) with respect to h^d yields

$$K_{\alpha\beta}\ q_{\beta}^{;d} = f_{\alpha}^{,d} - K_{\alpha\beta}^{,d}\ q_{\beta} \qquad (8)$$

To eliminate $q_{\alpha}^{;d}$ from (7), the adjoint system method is used. The adjoint vector λ_{α} may be defined so that the adjoint equations system takes the form

$$K_{\alpha\beta}\ \lambda_{\beta} = \Phi_{,\alpha} \qquad (9)$$

that, substituted (9) into (7) and on account of (8), implies

$$\Phi^{;d} = \Phi^{,d} + \lambda_{\alpha}(f_{\alpha}^{,d} - K_{\alpha\beta}^{,d}\ q_{\beta}) \qquad (10)$$

As mentioned above, the functions of random variables $K_{\alpha\beta}$, f_{α},, q_{β},, $K_{\alpha\beta}^{,d}$, $f_{\alpha}^{,d}$ and $\Phi^{,d}$ are now expanded around the expectations $\overline{b^r}$ via the second-order perturbation, with a given small parameter θ, symbolically written as

$$(\cdot)(\boldsymbol{h},\boldsymbol{b}) = (\cdot)^0 + \theta(\cdot)^{;r}\Delta b^r + \tfrac{1}{2}\theta^2(\cdot)^{;rs}\Delta b^r \Delta b^s \qquad r,s = 1,\dots,R \qquad (11)$$

in which Δb^r denotes the perturbational increment of br with respect to b_0^r , and $(\cdot)^0$, $(\cdot)^{;r}$ and $(\cdot)^{;rs}$ describe the zeroth, first and mixed (second) ordinary derivatives with respect to b^r.

Substituting the expansions of $K_{\alpha\beta}$, f_α, $\Phi_{,\alpha}$, q_β and λ_α into (2) and (9) and equating the coefficients of the parameter θ to zeroth, first and second power, we obtain

one pair of the zero-order equations

$$K_{\alpha\beta}^0 q_\beta^0 = f_\alpha^0 \tag{12a}$$

$$K_{\alpha\beta}^0 \lambda_\beta^0 = \Phi_{,\alpha}^0 \tag{12b}$$

r pair of the first-order equations

$$K_{\alpha\beta}^0 q_\beta^{;r} = f_\alpha^{;r} - K_{\alpha\beta}^0 q_\beta^0 \tag{13a}$$

$$K_{\alpha\beta}^0 \lambda_\beta^{;r} = \Phi_{,\alpha}^{;r} - K_{\alpha\beta}^0 \lambda_\beta^0 \qquad r = 1, \dots, R \tag{13b}$$

one pair of the second-order equations

$$K_{\alpha\beta}^0 q_\beta^{(2)} = \left(\frac{1}{2} f_\alpha^{;rs} - K_{\alpha\beta}^{;r} q_\beta^{;s} - K_{\alpha\beta}^{;rs} q_\beta^0\right) Cov(b^r, b^s) \tag{14a}$$

$$K_{\alpha\beta}^0 \lambda_\beta^{(2)} = \left(\frac{1}{2} \Phi_{,\alpha}^{;rs} - K_{\alpha\beta}^{;r} \lambda_\beta^{;s} - K_{\alpha\beta}^{;rs} \lambda_\beta^0\right) Cov(b^r, b^s) \tag{14b}$$

where

$$q_\alpha^{(2)} = \frac{1}{2} q_\alpha^{;rs} Cov(b^r, b^s) \tag{15a}$$

$$\lambda_\alpha^{(2)} = \frac{1}{2} \lambda_\alpha^{;rs} Cov(b^r, b^s) \qquad r, s = 1, \dots, R \tag{15b}$$

Having solved for q_β^0, $q_\beta^{;r}$ and $q_\beta^{(2)}$ in (12a), (13a) and (14a), the first two probabilistic moments for q_α can be computed by using the expansion (11) with $\theta = 1$, i.e.,

$$q_\alpha(\boldsymbol{h}, \boldsymbol{b}) = q_\alpha^0 + \theta q_\alpha^{;r} \Delta b^r + \frac{1}{2} \theta^2 q_\alpha^{;rs} \Delta b^r \Delta b^s \qquad r, s = 1, \dots, R \tag{16}$$

The expectation vector for q_α can then be obtained as

$$\overline{q_\alpha} = q_\alpha^0 + q_\alpha^{(2)} \tag{17}$$

To compute the cross-covariance matrix, we note that, cf. (16) and (17),

$$\Delta q_\alpha = q_\alpha - E[q_\alpha] = q_\alpha^{;r} \Delta b^r + \frac{1}{2} q_\alpha^{;rs} \Delta b^r \Delta b^s - q_\alpha^{(2)} \tag{18}$$

with $q_\alpha^{(2)}$ being a deterministic quantity so that $Cov(q_\alpha, q_\beta) = E[\Delta q_\alpha \Delta q_\beta]$ can be expressed as

$$Cov(q_\alpha, q_\beta) = q_\alpha^{;r} q_\beta^{;s} Cov(b^r, b^s) - q_\alpha^{(2)} q_\beta^{(2)} \tag{19}$$

It should be pointed out here that both the solutions (17) and (19) are second-order accurate, when compared with the 'conventional' ones, [6 - 8], in which only the expectation vector is second-order accurate, while the cross-covariance matrix is first-order accurate.

When q_α^0 and λ_α^0 from (12) are known, the functions $q_\alpha^{;r}$, $\lambda_\alpha^{;r}$, $q_\alpha^{(2)}$ and $\lambda_\alpha^{(2)}$can be solved by (13) and (14) in a sequential manner. In this way, it is possible to calculate the probability distribution of sensitivity. The expectations and cross-covariances of the sensitivity gradient are then written as, cf. [9]

$$\begin{aligned} \mathrm{E}[\Phi^{,d}] = \mathrm{G}^{0,d} + \frac{1}{2} G^{,d;rs} Cov(b^r, b^s) + A_\alpha^d \left(\lambda_\alpha^0 \lambda_\beta^{(2)}\right) + \\ -K_{\alpha\beta}^{0,d} q_\beta^{(2)} \lambda_\alpha^0 + (B_\alpha^{dr} \lambda_\alpha^{;s} + C_\alpha^{drs} \lambda_\alpha^0) Cov(b^r, b^s) \end{aligned} \tag{20}$$

and

$$\begin{aligned} Cov(\Phi^{,d}, \Phi^{,e}) = [\mathrm{G}^{,d;r} \mathrm{G}^{,e;s} + (\mathrm{G}^{,d;r} A_\alpha^e + \mathrm{G}^{,e;r} A_\alpha^d) \lambda_\alpha^{;s} + \\ +(\mathrm{G}^{,d;r} B_\alpha^{es} + \mathrm{G}^{,e;r} B_\alpha^{ds}) \lambda_\alpha^0 + A_\alpha^d A_\beta^e \lambda_\alpha^{;r} \lambda_\beta^{;s} + \\ +\left(A_\alpha^d B_\beta^{er} + A_\beta^e B_\alpha^{dr}\right) \lambda_\alpha^{;s} \lambda_\beta^0 + B_\alpha^{dr} B_\beta^{es} \lambda_\alpha^0 \lambda_\beta^0] \, Cov(b^r, b^s) \end{aligned} \tag{21}$$

where

$$A_\alpha^d \quad = f_\alpha^{0,d} - K_{\alpha\beta}^{0,d} q_\beta^0 \tag{22a}$$

$$B_\alpha^{dr} \quad = f_\alpha^{,d;r} - K_{\alpha\beta}^{0,d} q_\beta^{;r} - K_{\alpha\beta}^{,d;r} q_\beta^0 \tag{22b}$$

$$C_\alpha^{drs} = \frac{1}{2} f_\alpha^{,d;rs} - K_{\alpha\beta}^{,d;r} q_\beta^{;s} - \frac{1}{2} K_{\alpha\beta}^{,d;rs} q_\beta^0 \tag{22c}$$

with $d, e = 1, \dots, D$, $r, s = 1, \dots, R$; $\alpha, \beta = 1, \dots, N$. Also, the cross-covariance matrix (21) is obtained with the second-order accuracy, and not the first-order one as in [8, 9].

3 NUMERICAL RESULTS

This example (similar [11, 12]) the response of a thin shell structure is considered. Fig. 1 shows the half of a cylindrical shell clamped at boundaries under uniformly distributed $p = 100\ kN/m^2$ pressure (applied normal to shell surface). The remaining input data are: radius $R = 2{,}5\ m$, length $L = 12\ m$, Young modulus E= $30\ MPa$, Poisson ratio υ= 0,2. The expectation, correlation function and coefficient of variation of the shell thickness is assumed as:

$E(t) = t_0 = 0{,}06$ $\qquad R(t_r, t_s) = \vartheta \exp\{-abs\ [(x_r - x_0)(y_r - y_0)]/\lambda\}$ $\qquad \alpha = 0{,}05; 0{,}10; 0{,}15.$

where: $\vartheta = 1{,}5/RL$ $\qquad \lambda = 2{,}5RL$

The example was solved with the help of the program POLSAP [13]. Input data to example shows in Fig. 2.

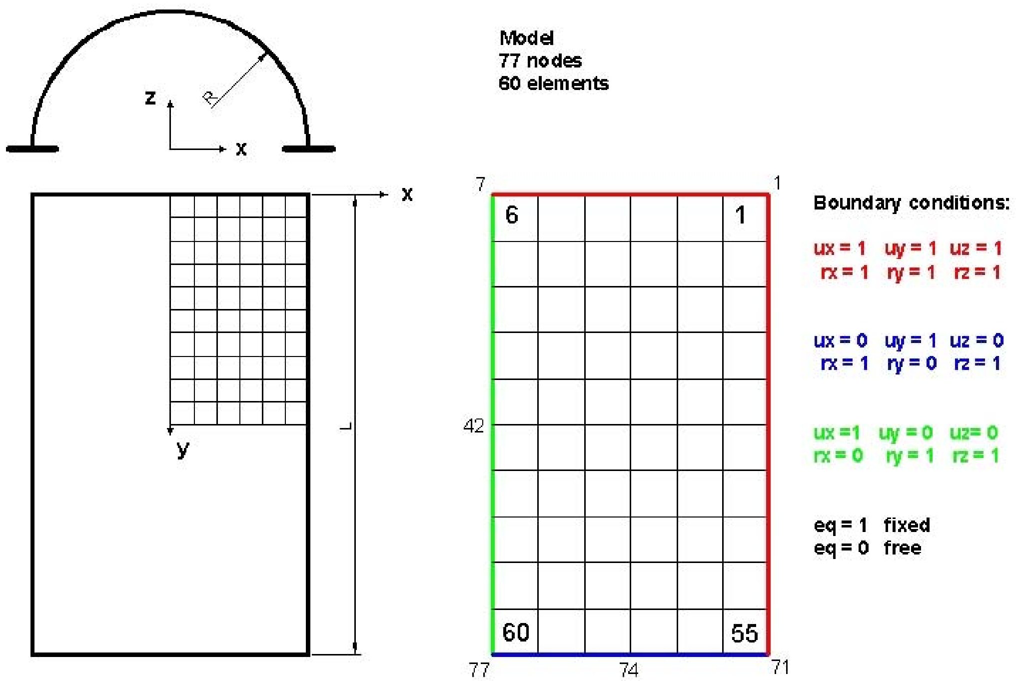

Fig. 1: Cylindrical shell with mesh grid

Due to symmetry only one-quarter of the shell is considered. The finite element mesh include 60 rectangular elements (60 random design variables), and the total number of degrees of freedom is 313. Tab. 1 and Fig. 3, Fig. 4 gives the computed values of the expectations and standard deviations of the sensitivity coefficient. The response functional is defined and considered at the mid-point (node 77) of the shell.

```
Cylindrical shell quarter: R=2.5m, L/2=6m
   77    1    1        12    0         0              1
    1C   1    1    1    1    1    1      2.5      0.0      90.0
   71C   1    1    1    1    1    1      2.5      6.0      90.0    7
    2C   1    1    1    1    1    1      2.5      0.0      75.0
    9C   0    0    0    0    0    0      2.5      1.0      75.0
   72C   0    1    0    1    0    1      2.5      6.0      75.0    7
    3C   1    1    1    1    1    1      2.5      0.0      60.0
   10C   0    0    0    0    0    0      2.5      1.0      60.0
   73C   0    1    0    1    0    1      2.5      6.0      60.0    7
    4C   1    1    1    1    1    1      2.5      0.0      45.0
   11C   0    0    0    0    0    0      2.5      1.0      45.0
   74C   0    1    0    1    0    1      2.5      6.0      45.0    7
    5C   1    1    1    1    1    1      2.5      0.0      30.0
   12C   0    0    0    0    0    0      2.5      1.0      30.0
   75C   0    1    0    1    0    1      2.5      6.0      30.0    7
    6C   1    1    1    1    1    1      2.5      0.0      15.0
   13C   0    0    0    0    0    0      2.5      1.0      15.0
   76C   0    1    0    1    0    1      2.5      6.0      15.0    7
    7C   1    1    1    1    1    1      2.5      0.0       0.0
   14C   1    0    0    0    1    1      2.5      1.0       0.0
   77C   1    1    0    1    1    1      2.5      6.0       0.0    7
    6   60    1                   1    1
         1
 3.1250d+7 6.2500d+6       0.0 3.1250d+7       0.0 1.2500d+6
        1.

    1    1    2    9    8         1    7      0.06       -100
   11    2    3   10    9         1    7      0.06       -100
   21    3    4   11   10         1    7      0.06       -100
   31    4    5   12   11         1    7      0.06       -100
   41    5    6   13   12         1    7      0.06       -100
   51    6    7   14   13         1    7      0.06       -100
   60   69   70   77   76         1           0.06       -100

   77    1      0.0       0.0      0.10

        1.
    1    1    0    0
```

Fig. 2: Listing input data to program POLSAP

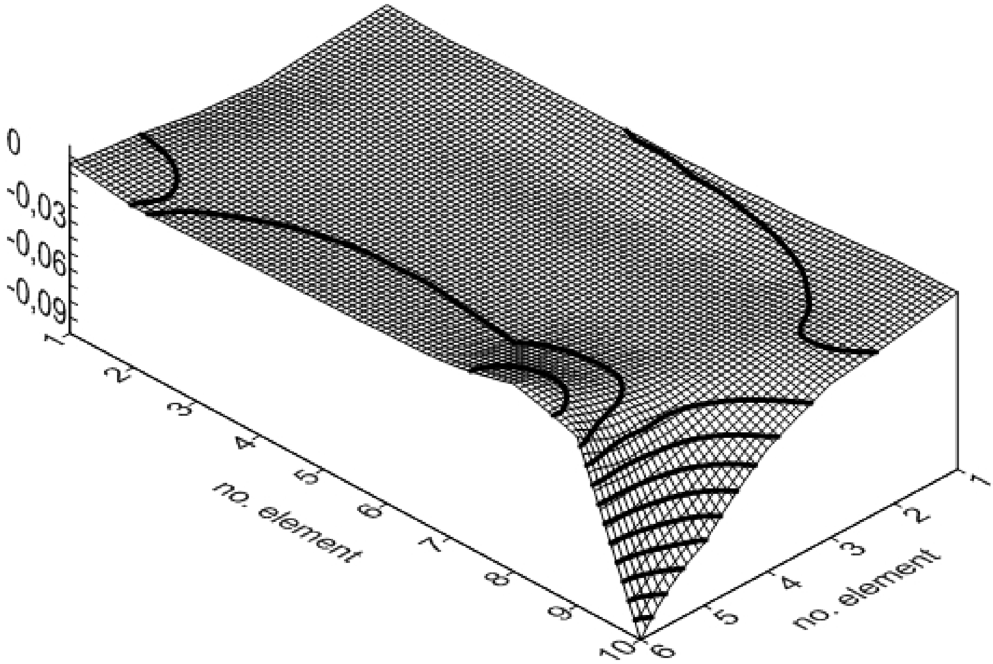

Fig. 3: Expectations of displacement sensitivity (node 77) - shell thicknesses as random variables

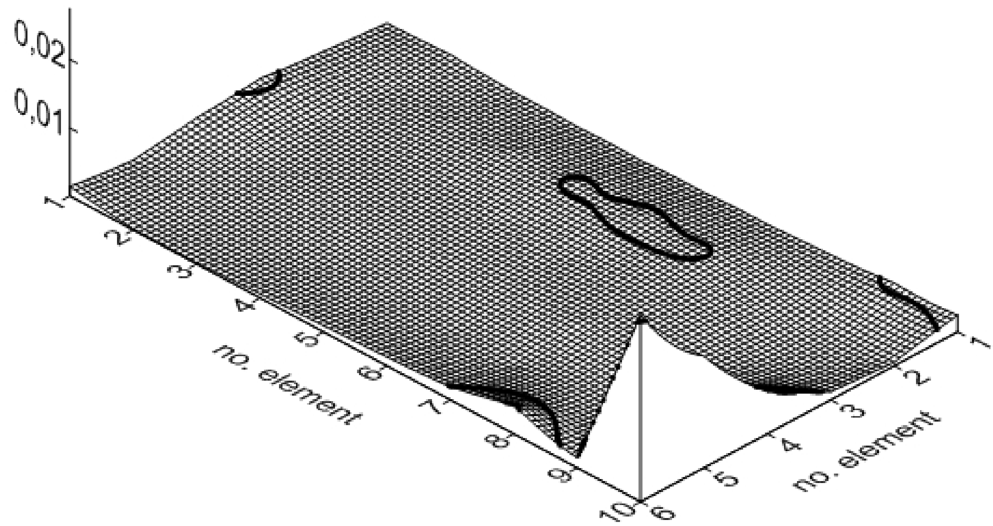

Fig. 4: Standard deviations of displacement sensitivity (node 77) - shell thicknesses as random variables

El.	Deterministic	Expectation			Std. Dev.		
		$\alpha = 0{,}05$	$\alpha = 0{,}10$	$\alpha = 0{,}15$	$\alpha = 0{,}05$	$\alpha = 0{,}10$	$\alpha = 0{,}15$
51	5,11E-03	5,15E-03	5,24E-03	5,40E-03	4,91E-04	9,81E-04	1,47E-03
52	-1,68E-04	-1,69E-04	-1,70E-04	-1,73E-04	8,31E-05	1,66E-04	2,49E-04
53	8,43E-04	8,49E-04	8,68E-04	8,99E-04	1,03E-04	2,06E-04	3,09E-04
54	2,56E-03	2,57E-03	2,62E-03	2,70E-03	2,31E-04	4,63E-04	6,94E-04
55	3,80E-03	3,82E-03	3,89E-03	4,01E-03	3,44E-04	6,89E-04	1,03E-03
56	5,18E-03	5,21E-03	5,31E-03	5,47E-03	4,71E-04	9,41E-04	1,41E-03
57	7,26E-03	7,30E-03	7,44E-03	7,68E-03	6,73E-04	1,35E-03	2,02E-03
58	1,57E-02	1,58E-02	1,61E-02	1,66E-02	1,45E-03	2,91E-03	4,36E-03
59	4,72E-03	4,75E-03	4,83E-03	4,98E-03	5,27E-04	1,05E-03	1,58E-03
60	-9,42E-02	-9,48E-02	-9,68E-02	-1,00E-01	9,13E-03	1,83E-02	2,74E-02

Tab. 1: Displacement design sensitivity - shell thickness as random design variables (only selected elements nearest for node 77), $[1/\mathrm{m}^2]$

4 CONCLUSIONS

We observe that some sensitivity values obtained here are negative. In general cases, in the static response of simple structures, the more massive system is, the lesser displacement is obtained. In this case, the displacement response is more complicated at places of the corner part, resulting in those negative values of sensitivity. This means that to decrease the displacement at the considered point 77, decreasing thickness of shell's elements in appropriate domains should be required in an alternative design point. The standard deviations of displacement sensitivity with respect to the shell thicknesses is about 10–30% of the expectations, Fig. 4. Comparing the stochastic and deterministic results, it can be concluded that they differ by no more than 3%, Tab. 1.

In paper ignores the discussion of the analysis of slenderness cylindrical shell in connection with stability and buckling. This can be realized in the future.

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